

INTRINSICALLY KNOTTED AND 4-LINKED DIRECTED GRAPHS

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ABSTRACT. We consider intrinsic linking and knotting in the context of directed graphs. We construct an example of a directed graph that contains a consistently oriented knotted cycle in every embedding. We also construct examples of intrinsically 3-linked and 4-linked directed graphs. We introduce two operations, consistent edge contraction and H-cyclic subcontraction, as special cases of minors for digraphs, and show that the property of having a linkless embedding is closed under these operations. We analyze the relationship between the number of distinct knots and links in an undirected graph G and its corresponding symmetric digraph \overline{DG} . Finally, we note that the maximum number of edges for a graph that is not intrinsically linked is $O(n)$ in the undirected case, but $O(n^2)$ for directed graphs.

1. INTRODUCTION

An abstract graph G is called *intrinsically linked* (*intrinsically knotted*) if every embedding of that graph into three-space contains a nontrivial link (knot). Intrinsically linked and intrinsically knotted graphs were first studied in the early 1980s by Sachs [17] and by Conway and Gordon [2]. In this paper, we will consider directed graphs and use the terms digraph and directed graph interchangeably. Undirected graphs will be denoted G , and directed graphs as \overline{G} . For a graph G , the digraph \overline{DG} is the symmetric digraph corresponding to G formed by replacing each edge v_1v_2 of G with the directed edges v_1v_2 and v_2v_1 .

In [11], Foisy, Howards and Rich consider intrinsic linking in the context of directed graphs, defining a directed graph \overline{G} to be intrinsically linked (knotted) if every embedding of the digraph \overline{G} contains a nontrivial link (knot) where every cycle in the link (knot) has a consistent orientation. This definition is motivated by applications where the graph represents a flow, such as an electrical current, and the edges of the graph must have a consistent orientation for the flow to occur. Foisy, Howards and Rich then prove that the complete symmetric digraph on six vertices \overline{DK}_6 is intrinsically linked. We continue their work by finding an example of an intrinsically knotted directed graph in Section 3.

Variations of intrinsic linking have been studied that require more complex structures in every embedding of the graph, such as a nonsplit link of n components [3], a link where one or more components are non-trivial knots [7], or arbitrary linking patterns between the components [5]. We begin extending the theory of intrinsic linking for directed graphs in this direction by demonstrating examples of intrinsically 3- and 4-linked directed graphs in Sections 4 and 5.

Researchers have also studied the minimal number of links (knots) that are contained in the embeddings of an intrinsically linked (knotted) graph [8], [1], see also [4] for a summary of similar results when restricting to straight line embeddings

of such graphs. Inspired by a question of [11], we investigate the number of distinct knots or links in directed graphs of the form \overline{DG} . We construct bounds on the minimum number of knots or links in \overline{DG} based on the minimum number of knots or links in G , and show that if \overline{DG} is intrinsically linked, then it must contain at least 4 distinct, consistently oriented links. This is in contrast to arbitrary directed graphs, and we provide an example of an intrinsically linked directed graph that admits an embedding with a single non-trivial consistently oriented link.

For undirected graphs, the property of having a linkless or knotless embedding is closed under the operation of taking graph minors [15], and the minor minimal family for intrinsically linked graphs was characterized by Robertson, Seymour and Thomas [16]. In a sharp contrast, Foisy, Howards and Rich [11] show that this is not the case for intrinsic linking in directed graphs— a directed graph that has an embedding with no non-split consistently oriented link may have a minor that is intrinsically linked. In Section 2, we examine the operation of vertex expansion, and answer a question of [11]. We then introduce the notions *consistent edge contraction* and *H-cyclic subcontraction*. Consistent edge contraction preserves the property of having a linkless embedding for digraphs, and H-cyclic subcontraction does as well under mild assumptions. Consistent edge contraction is used for some of our proofs, but further investigation is needed to determine the best analogue of minors for studying intrinsically linked directed graphs, and whether any of the possible operations will lead to a finite forbidden set for intrinsically linked directed graphs.

Finally, in Section 8 we note an additional difference between intrinsic linking in the directed and undirected cases. We consider the extremal problem of the maximum number of edges for a graph G or a digraph \overline{G} on n vertices to admit a linkless embedding. For undirected graphs, there is a constant c such that any graph on n vertices with more than cn edges contains a K_6 minor [14] [19], hence is intrinsically linked. However for undirected graphs, as noted in [11], there are examples of directed graphs with $O(n^2)$ edges that are not intrinsically linked. Thus, as n becomes large, we should expect many directed graphs to be intrinsically linked as undirected graphs (that is, contain a non-split link in every embedding when ignoring edge orientations), but not be intrinsically linked as directed graphs (i.e. admit an embedding where all consistently oriented cycles form only split links). A similar result holds for intrinsic knotting.

2. VERTEX EXPANSION AND GRAPH MINORS

We first consider vertex expansion in digraphs, and then turn our attention to operations that are variations of edge contraction. In [11], Question 5.1 asks if conducting a single vertex expansion of an intrinsically linked directed graph preserves the property of intrinsic linking for some choice of orientation for the new edge. We demonstrate the answer to this question is “no” by providing a counterexample.

Theorem 2.1. *Let \overline{G} be the directed graph in Figure 1, and \overline{G}' and \overline{G}'' the directed graphs obtained by vertex expansion at a_2 as shown in Figure 2. Then \overline{G} contains a consistently oriented non-split link in every spatial embedding, but both \overline{G}' and \overline{G}'' have embeddings with no non-split consistently oriented link.*

Proof. We can see that the directed graph \overline{G} is intrinsically linked as follows. For a given embedding of \overline{G} , when ignoring the edges from vertex set B to A , we may

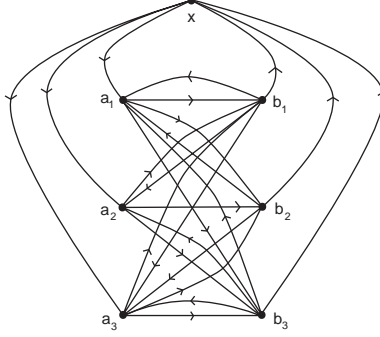


FIGURE 1. The digraph \overline{G} for Theorem 2.1. Note the single edge from a_2 to b_2 .

find a triangle T and a square S with odd linking number, as $K_{3,3,1}$ is a minor of this graph. Note that by construction, T is consistently oriented. If S is not consistently oriented, we may construct a consistently oriented cycle that has odd linking number with T using the edges directed from B to A and the techniques described in the proof of Theorem 3.5 of [11]. If S contains the edge a_2b_2 , this merely forces the orientation for the construction.

We now show that $\overline{G'}$ and $\overline{G''}$ admit linkless embeddings. For simplicity, it will suffice to consider embeddings where all bigons bound disks. There is an embedding of $K_{3,3,1}$ that contains a single nontrivial link, consisting of a triangle and square [8], and by symmetry, every triangle-square partition is the lone non-trivial link in some embedding. Let f be an embedding of $K_{3,3,1}$ where the sole non-trivial link is $(x, a_2, b_3)(a_1, b_1, a_3, b_2)$. This gives an embedding of \overline{G} , which we will call f by abuse of notation, where the only nontrivial consistently oriented links are $(x, a_2, b_3)(a_1, b_1, a_3, b_2)$ and $(x, a_2, b_3)(a_1, b_2, a_3, b_1)$. Let f' be an embedding of $K_{3,3,1}$ where the sole non-trivial link is $(x, a_1, b_1)(a_2, b_2, a_3, b_3)$. Due to the single edge from a_2 to b_2 in \overline{G} , the embedding f' of \overline{G} contains a single consistently oriented non-trivial link $(x, a_1, b_1)(a_2, b_2, a_3, b_3)$.

The graphs $\overline{G'}$ and $\overline{G''}$ are formed from \overline{G} by expansion of vertex a_2 . By shrinking the edge e' or e'' , we may cause vertices a'_2 and a''_2 and edge e' (or e'') to lie in a neighborhood of vertex a_2 , so we may extend the embeddings f and f' to $\overline{G'}$ and $\overline{G''}$. Now for the digraph $\overline{G'}$, the cycle $(a'_2, a'_2, b_2, a_3, b_3,)$ is inconsistently oriented. Suppose that the embedding f' of $\overline{G'}$ contained a consistently oriented non-split link that did not use the edge e' , then embedding f' of \overline{G} would contain more than one consistently oriented non-split link, a contradiction. If embedding f' of $\overline{G'}$ contains a consistently oriented non-split link that does contain e' , then that link is preserved by contracting e' , again creating a second, distinct non-split link in embedding f' of \overline{G} . Thus, the embedding f' of $\overline{G'}$ has no consistently oriented link. Similarly, for $\overline{G''}$, the cycle (x, a'_2, a''_2, b_3) is inconsistently oriented, so the embedding f of $\overline{G''}$ has no consistently oriented link. \square

Foisy-Howard-Rich noted in Theorem 4.2 of [11] that standard vertex expansion does not preserve intrinsic linking in directed graphs, and Theorem 2.1 shows that

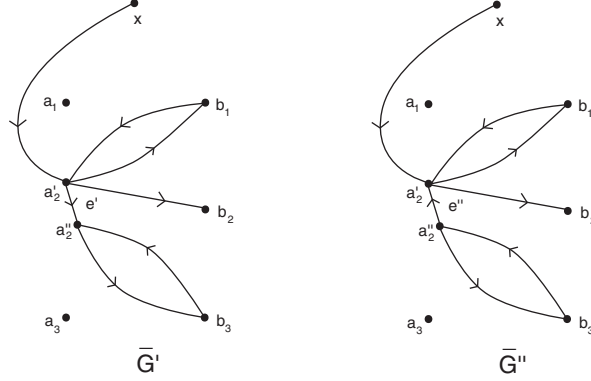


FIGURE 2. The edges adjacent to a'_2 and a''_2 in $\overline{G'}$ and $\overline{G''}$.

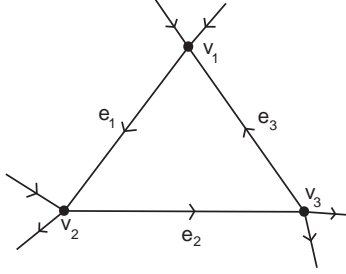


FIGURE 3. The edges e_1 and e_2 are candidates for consistent edge contraction, but edge e_3 is not. Note also that the vertices v_1, v_2, v_3 can be replaced by a single vertex using H-cyclic subcontraction, but no subset of them can be H-cyclicly subcontracted.

controlling the edge orientation in arbitrary vertex expansion is also insufficient. We introduce two operations to address these challenges: *consistent edge contraction* and *H-cyclic subcontraction*. We describe these each in turn.

Let $\overline{G'}$ be the digraph obtained from \overline{G} by splitting a vertex v into vertices v_1 and v_2 , and adding an edge e directed from v_1 to v_2 . If v_1 is a sink in $\overline{G'} \setminus e$ or v_2 is a source in $\overline{G'} \setminus e$, we will say $\overline{G'}$ is obtained from \overline{G} by a *consistent vertex expansion*. Similarly, if \overline{G} is obtained from $\overline{G'}$ by contracting such an edge, we will say \overline{G} is obtained by a *consistent edge contraction*. See Figure 3.

Proposition 2.2. *Let \overline{G} and $\overline{G'}$ be directed graphs. If \overline{G} is intrinsically linked and $\overline{G'}$ is obtained from \overline{G} by a consistent vertex expansion, then $\overline{G'}$ is intrinsically linked as a digraph. Equivalently, if $\overline{G'}$ has a linkless embedding, and \overline{G} is obtained from $\overline{G'}$ by a consistent edge contraction, then \overline{G} has a linkless embedding. That is, the property of having a linkless embedding is closed under consistent edge contraction.*

Proof. There is a one-to-one correspondence of consistently oriented cycles in \overline{G} and $\overline{G'}$ under these moves, so we may proceed as in the undirected case.

In any embedding f' of $\overline{G'}$, we may shrink the edge e by isotopy until it is contained a ball disjoint from the rest of $\overline{G'}$. This gives an embedding f of \overline{G} . If f contains a consistently oriented non-split link, then isotopic consistently oriented cycles can be found in f' , so it contains a link as well. \square

We note that the operation of taking minors of directed graphs described by [11] corresponds to the *weak subcontraction* of Jagger [13]. Jagger also defines *strong subcontraction* as a generalization of undirected graph minors to directed graphs. If \overline{H} is obtained from \overline{G} by strong subcontraction, any consistently oriented cycle of \overline{H} corresponds to at least one consistently oriented cycle of \overline{G} . Hence, strong subcontraction avoids the problems highlighted by Theorem 2.1, as neither $\overline{G'}$ nor $\overline{G''}$ strongly subcontracts to \overline{G} . Strong subcontraction is defined combinatorially and generally does not have a clear geometric interpretation, but it may be a promising technique for studying intrinsically linked directed graphs. We will define an operation we call *H-cyclic subcontraction* that is more restrictive than strong subcontraction, but easier to understand geometrically.

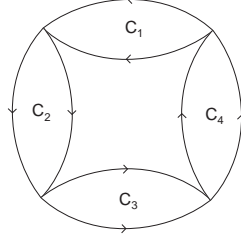
First we review some definitions. Recall that a digraph \overline{G} is strongly connected if for any pair of vertices v, w in \overline{G} , there exists a directed path from v to w . Let \overline{H} be a digraph with vertex set $\{w_0, w_1 \dots w_r\}$ and $V_0, V_1, \dots V_r$ a partition of the vertices of \overline{G} such that if $w_i w_j$ is an edge of \overline{H} there exist $v_{ik} \in V_i$ and $v_{jl} \in V_j$ such that $v_{ik} v_{jl}$ is an edge of \overline{G} . Then \overline{H} is a minor of \overline{G} if the induced subgraph $G(V_i)$ is connected as an undirected graph for all V_i (that is, weakly connected as a digraph). The graph \overline{H} is a strong subcontraction (or strong minor) of \overline{G} if the induced subgraph $\overline{G(V_i)}$ is strongly connected as a digraph for all V_i . We define \overline{H} to be an H-cyclic minor (or H-cyclic subcontraction) if the induced subgraph $\overline{G(V_i)}$ contains a consistently oriented Hamiltonian cycle for all V_i . Note that contracting a symmetric pair of edges is a special case of H-cyclic subcontraction.

Proposition 2.3. *Let \overline{H} be an H-cyclic minor of a digraph \overline{G} . Then if \overline{G} has an embedding where no pair of consistently oriented cycles have non-zero linking number, then \overline{H} has an embedding where no pair of consistently oriented cycles have non-zero linking number.*

Proof. If there is more than one partition V_i of \overline{G} that contains more than one vertex, we may proceed sequentially. Thus, we can assume $V_0 = \{v_{01} \dots v_{0n}\}$ and $V_i = v_{i0}$ for all $i \neq 0$. Let f be an embedding of \overline{G} with no link with non-zero linking number. Let C_0 be the consistently oriented Hamiltonian cycle of $\overline{G(V_0)}$. We form an embedding of \overline{H} as follows: in $f(G)$, delete one edge of C_0 to form a consistent path P_0 that includes all vertices of V_0 . Delete all edges in $\overline{G(V_0)} \setminus P_0$. Contract P_0 . This gives an embedding f of \overline{H} .

If $f(\overline{H})$ has a link L_1, L_2 with nonzero linking number that does not use w_0 , then this link is contained in $f(\overline{G})$, a contradiction. So we may assume that $w_0 \in L_1$. By construction, there are vertices v_{0i} and v_{0j} such that L_1 corresponds to a path P_1 in $f(\overline{G} \setminus \overline{G(V_0)})$ that connects v_{0i} and v_{0j} . Let P_{ij} be the subpath of P_0 that connects v_{0i} and v_{0j} . Then $P_1 \cup P_{ij}$ in $f(\overline{G})$ is isotopic to L_1 . If $P_1 \cup P_{ij}$ is consistently oriented, then $lk(P_1 \cup P_{ij}, L_2) \neq 0$, a contradiction.

If $P_1 \cup P_{ij}$ is not consistently oriented, we may form $P_{ji} = C_0 \setminus P_{ij}$. Then $P_1 \cup P_{ji}$ is consistently oriented, as is C_0 . Since $C_0, P_1 \cup P_{ji}$ and $P_1 \cup P_{ij}$ divide S^2 into

FIGURE 4. The counterclockwise oriented $\overline{D_4}$.

regions, and L_2 has non-zero linking number with $P_1 \cup P_{ij}$, L_2 must have non-zero linking number with either C_0 or $P_1 \cup P_{ji}$. This gives a consistently oriented link with non-zero linking number in $f(\overline{G})$, a contradiction. \square

Note that every undirected graph that is intrinsically linked contains a link with non-zero linking number, due to the classification of forbidden minors for intrinsically linked graphs [16]. If every embedding of an intrinsically linked digraph must contain a link with non-zero linking number, then Proposition 2.3 shows that the property of having a linkless embedding is closed under H-cyclic subcontraction. A similar caveat is necessary for the result claimed in Theorem 4.3 of [11].

3. AN INTRINSICALLY KNOTTED DIRECTED GRAPH

The following is a direct corollary of a result of Taniyama-Yasuhara [18], and independently Foisy [9], applied to the directed graph case. Let D_4 be the graph pictured in Figure 4.

Corollary 3.1. *Let $\overline{D_4}$ be an oriented D_4 graph, with all edges oriented in the counterclockwise direction, or all edges oriented in the clockwise direction. If $lk(C_1, C_3) \neq 0$ modulo 2 and $lk(C_2, C_4) \neq 0$ modulo 2, then $\overline{D_4}$ contains a consistently oriented Hamiltonian cycle that is a non-trivial knot.*

Proof. Ignoring edge orientations, the Taniyama-Yasuhara/Foisy result implies that there is a Hamiltonian cycle in $\overline{D_4}$ with odd Arf invariant. As all Hamiltonian cycles of $\overline{D_4}$ are consistently oriented, we have the result. \square

If every embedding of a digraph \overline{G} can be reduced to a copy of $\overline{D_4}$ via edge deletions and consistent edge contractions (in the sense of Section 2), and this copy of $\overline{D_4}$ satisfies the conditions of Corollary 3.1, then every embedding of \overline{G} contains a consistently oriented cycle that is a non-trivial knot. That is, \overline{G} is intrinsically knotted as a digraph.

We now produce an example of a consistently oriented intrinsically knotted digraph \overline{G} on 11 vertices. We construct \overline{G} from two vertex sets, A of five vertices $(a_1, a_2, a_3, a_4, a_5)$ and B of four (b_1, b_2, b_3, b_4) , and 2 distinguished vertices v and w as follows. Add edges so that $a_1, a_2, a_3, b_1, b_2, b_3$ form a $K_{3,3}$ with all edges directed from a_i to b_j . Add edges from a_1, a_2, a_3 to v and edges from v to b_1, b_2, b_3 . Add a 3 cycle of arbitrary orientation between b_1, b_2, b_3 . Add 3 edges, directed from each b_i to b_4 . Add edges from each b_i to each a_j . Add an edge from each b_i to w and

from w to each a_j . Add edges so that a_1, a_2, a_3 form a $\overline{DK_3}$. Add an edge between a_4, a_5 and from each of a_4 and a_5 to each of a_1, a_2, a_3 .

Theorem 3.2. *Every embedding of \overline{G} contains a consistently oriented cycle that is a non-trivial knot.*

Proof. Within any embedding of \overline{G} we can identify a copy of $\overline{D_4}$ that satisfies the conditions of Corollary 3.1 as follows. The vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and v with the edges directed from A to B form $K_{3,3,1}$. We may find a triangle-square pair with odd linking number. Suppose the triangle T_1 is a_1, v, b_1 . The square is a_2, b_2, a_3, b_3 . Using the edge b_2b_3 we find a triangle T_3 that has odd linking number with T_1 . T_3 contains b_2, b_3 and one of a_2, a_3 . Say a_2 . Note that a_1 and a_2 are sources. The vertex b_1 is a sink, and so is one of b_2, b_3 . Say it is b_2 .

Now, $b_1, b_2, b_4, a_3, a_4, a_5$ and w along with the edges directed from B to A form a $K_{3,3,1}$. (This does not depend on the choice of vertex labels in the previous paragraph). We may similarly find two triangles T_2 and T_4 with odd linking number.

We now construct the $\overline{D_4}$. If the source vertices of T_2 and T_4 are b_1 and b_2 , they are the sink vertices of T_1 and T_3 and we continue. If T_4 has source vertex b_4 instead of b_2 we add the edge b_2b_4 . T_2 and T_4 have sink vertices in $\{a_3, a_4, a_5\}$. These are disjoint from the source vertices of T_1 and T_3 , (a_1 and a_2). As each of a_3, a_4, a_5 has an edge directed to each of a_1 and a_2 , we can add the edges needed to get the adjacencies required to complete the $\overline{D_4}$.

Note that the other vertices contained in T_1, T_3 (v and b_2) are disjoint from T_2, T_4 , and the other vertices of T_2, T_4 (w and one of a_3, a_4, a_5) are disjoint from T_1, T_3 . \square

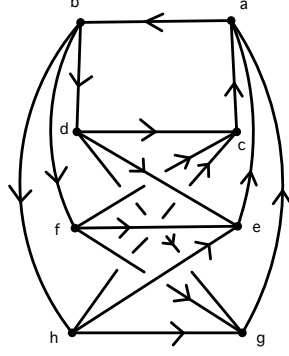
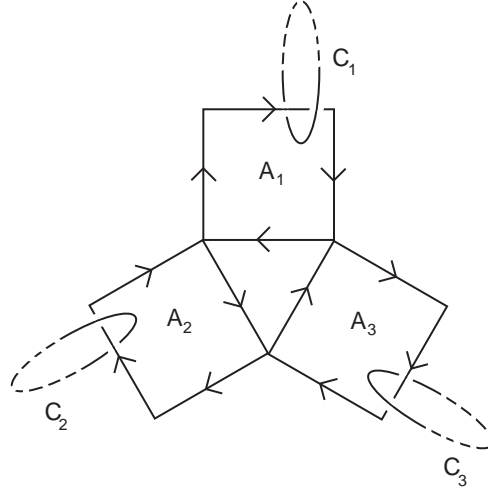
4. CONSTRUCTING A 3-LINKED DIRECTED GRAPH

We include this construction of an intrinsically 3-linked directed graph both to demonstrate key ideas for the 4-linked construction and to provide a bound for Question 5.5 of [11]. Question 5.5 asks for the minimal n such that the symmetric complete digraph $\overline{DK_n}$ is intrinsically 3-linked. The example below shows that $n \leq 21$. As K_9 can be embedded with no triple link [6], and any embedding of K_n with no 3-link can be extended to an embedding of $\overline{DK_n}$ with no 3-link (by embedding the symmetric edges so that they bound disks), we have $10 \leq n \leq 21$.

Lemma 4.1. *Let \overline{H} be the directed graph whose underlying graph is $K_{4,4}$, with two partitions $\{b, c, e, g\}$, and $\{a, d, f, h\}$, and every edge between $\{d, f, h\}$ and $\{c, e, g\}$ directed towards $\{c, e, g\}$, every edge from b to $\{d, f, h\}$ directed towards $\{d, f, h\}$, and every edge between $\{c, e, g\}$ and a directed towards a , and finally the edge between a and b directed towards b . Then the edge $a - b$ is contained in a consistently oriented link with non-zero linking number in every embedding of \overline{H} .*

Proof. Sachs [17] showed that given any edge of $K_{4,4}$ and any spatial embedding of $K_{4,4}$, that edge is contained in a non-split link. By construction of \overline{H} , the edge $a - b$ is only contained in consistently oriented cycles. The result then follows. \square

Theorem 4.2. *Let \overline{H} be as above, and \overline{G} the graph formed from 3 copies of \overline{H} by identifying the end points of the preferred edges $a - b$ so that they form a consistently oriented triangle. Then every embedding of \overline{G} contains a non-split consistently oriented 3-link.*

FIGURE 5. The digraph \overline{H} .FIGURE 6. Gluing three copies of \overline{H} .

Proof. The graph \overline{G} is formed from three copies of \overline{H} , label them \overline{H}_i . Embed \overline{G} . Let A_i be the cycles containing a_i and b_i in the non-split, consistently oriented link in \overline{H}_i . Let C_i be the cycle in \overline{H}_i that has non-zero linking number with A_i . See Figure 6. Let Z be the consistent 3-cycle formed from the edges $a_i - b_i$, and let W be the consistent 9-cycle formed from the other edges of the A_i . (Z is the “inside” of the ring, W is the “outside”). Then A_i, Z, W are a division of S^2 into 5 consistent cycles, and hence each of the C_i must have non-zero linking number with at least two of these cycles. By the pigeonhole principle, two of the C_i have non-zero linking number with the same cycle, giving a 3-link. \square

5. CONSTRUCTING A 4-LINKED DIRECTED GRAPH

Let \overline{H} be the digraph in Figure 5. We will use the digraph \overline{H} as the building block for the construction of the 4-linked directed graph, and Lemma 4.1 will be a key tool in the proof.

Lemma 5.1. *Let \overline{G} be the digraph formed from 11 copies of \overline{H} , by identifying a_1 with b_2 , a_2 with b_3 and so on to a_{11} identified to b_1 . Then every embedding of \overline{G} contains either a consistently oriented non-split 4-link or a set of three non-split 3-links that have no common cycle.*

Proof. Embed \overline{G} , and assume that the embedding does not contain a consistently oriented non-split 4-link. Let A_i be the cycles containing a_i and b_i in the non-split, consistently oriented link in \overline{H}_i , and let C_i be the other components in those links. Let Z be the consistent 11-cycle formed from the edges $a_i - b_i$, and let W be the consistent 33-cycle formed from the other edges of the A_i . (Z is the “inside” of the ring, W is the “outside”). Then A_i, Z, W are a division of S^2 into 13 consistent cycles, and thus each of the 11 C_i must have non-zero linking number with at least 2 of these cycles.

As there is no 4-link, at most 2 of the C_i can have non-zero linking number with Z , and at most 2 can non-zero linking number with W . Thus at least 7 of the C_i must have non-zero linking number with A_j where $i \neq j$. As there is no 4-link, for each A_i there is at most one C_j , $i \neq j$ that has non-zero linking number with A_i . Thus the set $A = \{A_i | lk(A_i, C_j) \neq 0 \text{ for } i \neq j\}$ has at least 7 members. Choose a member of A , call it A_1 . It has non-zero linking number with C_1 from \overline{H}_1 and a second C_j , call it C_2 . Remove C_1 and C_2 from consideration. This leaves at least 5 C_i that have non-zero linking number with A_j where $i \neq j$. Thus A must have at least 5 members remaining, and we may choose A_3 in A , with $A_3 \neq A_2$. A_3 has non-zero linking number with C_3 from \overline{H}_3 and an additional cycle C_4 . As before, removing C_3 and C_4 from consideration leaves at least 3 of the C_i with non-zero linking number with A_j where $i \neq j$. Thus A must have at least 3 members, and we may choose $A_5 \in A$ with $A_5 \neq A_2, A_4$. The cycle A_5 has non-zero linking number with C_5 and C_6 . So we have three 3-links $A_1, C_1, C_2, A_3, C_3, C_4$, and A_5, C_5, C_6 , with all of the C_j, A_i distinct (though the A_i may share vertices). \square

Theorem 5.2. *Let \overline{H} be as above, and \overline{G} the digraph formed from $2 \binom{11}{2} - 11$ copies of \overline{H} by identifying the a_i and b_j such that the edges $a_i b_i$ form $\overline{DK}_{11} \setminus B$, where B is a consistently oriented Hamiltonian cycle. Then \overline{G} contains a non-split consistently oriented 4 component link in any spatial embedding.*

Proof. Embed \overline{G} , and let \overline{G}' denote the subgraph of \overline{G} made up of the 11 copies of \overline{H} whose edges $a_i - b_i$ make up the unique consistently oriented Hamiltonian cycle of \overline{G} that corresponds to the missing Hamiltonian cycle B (with opposite orientation). If \overline{G}' does not contain a 4-link, then by Lemma 5.1, it must contain three distinct 3-links. Renumbering the \overline{H}_i as necessary, we may call the components of these links A_2, C_{21}, C_{22} ; A_4, C_{41}, C_{42} , and A_6, C_{61}, C_{62} , where A_i is the component containing the edge $a_i - b_i$, and C_{ij} the cycles that have non-zero linking number with A_i .

If A_2, A_4 and A_6 do not share any vertices, we use the copies of \overline{H} that make up the rest of \overline{G} to form a new subgraph \overline{G}'' of \overline{G} as follows. Let \overline{H}_1 be the copy of \overline{H} where a_1 is identified with b_6 and b_1 with a_2 . \overline{H}_3 is the graph with a_3 identified

with b_2 and b_3 with a_4 , and similarly for \overline{H}_5 . Call the components of the 2-links in \overline{H}_i : A_i and C_{i1} (for i odd).

Then we may form a consistent 6-cycle Z from the edges $a_i - b_i$ of $\overline{G''}$, and an 18-cycle W from the other edges of the A_i . The cycles A_i , Z and W divide S^2 into consistently oriented regions, so each C_{ij} must have non-zero linking number with at least 2 cycles in $A = \{A_i, Z, W\}$. There are 9 C_{ij} and 8 cycles in A . Each C_{ij} must have non-zero linking number with at least 2 cycles in A , and as $\frac{2 \cdot 9}{8} > 2$, some cycle in A must have non-zero linking number with at least 3 of the C_{ij} . This gives the 4-link. (Alternately, we can apply Lemma 6.2 below, and as $6 > 2(3-1)$, $\overline{G''}$ must contain a 4-link.)

If the some of A_2 , A_4 , A_6 share a vertex, we may repeat the above argument omitting one (or more) of the “bridging” copies of \overline{H} in the construction of $\overline{G''}$. \square

6. TOWARDS AN N-LINKED DIRECTED GRAPH

We expect the construction of an intrinsically n -linked directed graph for $n > 4$ to be possible, and include the following lemmas. Each may serve as the final step in an inductive argument constructing such a graph.

Lemma 6.1. *If c_1 is a consistently oriented cycle in \overline{DG} and c_1 has non-zero linking number with n disjoint arbitrary cycles, then c_1 has non-zero linking number with n disjoint consistently oriented cycles.*

Proof. If $lk(c_1, c_j) \neq 0$ then either $lk(c_1, \overline{v_i v_j} \overline{v_j v_i}) \neq 0$ for some adjacent vertices v_i, v_j , in c_j , or c_j can be modified to a consistent cycle c'_j by replacing any inconsistent edge $\overline{v_i v_j}$ with the consistent edge $\overline{v_j v_i}$ without changing $lk(c_1, c_j)$. As only the cycle c_j is modified, $lk(c_1, c_i)$ is unaffected for $i \neq j$. \square

Lemma 6.2. *Let \overline{H} be a digraph such that in every embedding of \overline{H} , a fixed edge is contained a consistent cycle that has non-zero linking number with $n-1$ disjoint, consistently oriented cycles of \overline{H} . Let $\overline{H'}$ be a digraph for which every embedding contains a 2 link with non-zero linking number, and that link uses a fixed edge. Let k be even, and \overline{G} a graph formed from $\frac{k}{2}$ copies of \overline{H} and $\frac{k}{2}$ copies of $\overline{H'}$ so that the preferred edges form a consistent k cycle. Then every embedding of \overline{G} contains a consistently oriented non-split $n+1$ link when $k > 2(n-1)$.*

Proof. Let the components of the links in \overline{H} and $\overline{H'}$ that use the preferred edge be denoted as A_i , and the other components of those links as C_{ij} . Let Z be the consistently oriented k cycle formed by the preferred edges, and W the (consistent) cycle formed by the other edges of the A_i . Then A_i , Z , W divide S^2 into $k+2$ consistently oriented regions.

There are $\frac{k}{2}n$ cycles C_{ij} , and each of these must have non-zero linking number with at least two regions. Thus, the C_{ij} have non-zero linking number with at least kn regions in total.

Some region has non-zero linking number with at least n of the C_{ij} when $\frac{kn}{k+2} > n-1$. Thus \overline{G} must contain an $n+1$ link when $k > 2(n-1)$. \square

7. COUNTING LINKS AND KNOTS IN DIRECTED GRAPHS

Recall that for a graph G , the digraph \overline{DG} is the symmetric digraph corresponding to G formed by replacing each edge $v_1 v_2$ of G with the directed edges $v_1 v_2$

and v_2v_1 . In [11], Question 5.2 asks if G is intrinsically linked, what is the minimum number of distinct links in any embedding of \overline{DG} ? We provide some general bounds and a precise answer for the case when G has an embedding with a single non-trivial link.

We will follow notation from [8] and [1] and let $mnl(G)$ denote the minimal number of distinct, non-split links (of any number of components) over all embeddings of G , and $\overline{mnl}(\overline{G})$ the minimal number of distinct, consistently oriented nonsplit links over all embeddings of a directed graph \overline{G} . Let $mnl_n(G)$ denote the minimal number of distinct non-split n component links in any embedding of a graph G , and $\overline{mnl}_n(\overline{G})$ denote the minimal number of distinct consistently oriented non-split n component links in any embedding of a directed graph \overline{G} . Let $mnk(G)$ denote the minimum number of nontrivial knots over all embeddings of G , and $\overline{mnk}(\overline{G})$ be the minimum number of consistently oriented nontrivial knots in any embedding of a directed graph \overline{G} . Let $\Gamma(G)$ denote the set of all cycles in G .

Proposition 7.1. *For a graph G , $\overline{mnk}(\overline{DG}) \leq 2 * mnk(G)$.*

Proof. Let f be an embedding of G that realizes $mnk(G)$. We may extend f to an embedding of \overline{DG} by thickening the edges of $f(G)$ and embedding \overline{DG} so that the bigons formed by the edges corresponding to e bound a disk within that thickened edge. Then for a knot K in $f(G)$ there are two isotopic, consistently oriented cycles, K^+ and K^- , in $f(\overline{DG})$. Any other consistently oriented cycle C in $f(\overline{DG})$ must be a trivial knot, as if $C \in \Gamma(f(G))$ when forgetting edge orientations, then C is trivial. If $C \notin \Gamma(f(G))$ when forgetting edge orientations, then C is a bigon corresponding to an edge e , and hence bounds a disk by the construction of $f(\overline{DG})$. Thus $f(\overline{DG})$ contains exactly $2 * mnk(G)$ nontrivial knots, and we have the bound. \square

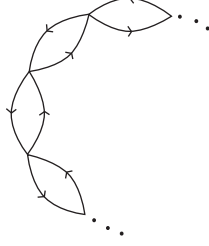
Lemma 7.2. *Let $f(G)$ be an embedding of a spatial graph G that contains k_n non-split n component links. Then $\overline{mnl}_n(\overline{DG}) \leq k_n * 2^n$ and $\overline{mnl}(\overline{DG}) \leq \sum_n k_n * 2^n$.*

Proof. We may extend $f(G)$ to an embedding of \overline{DG} by thickening the edges of $f(G)$ and then embedding \overline{DG} so that the bigon formed by the directed edges corresponding to an edge e bound a disk within the thickened edge $f(e)$. Call this embedding $f(\overline{DG})$.

We now count the distinct non-split n component links in $f(\overline{DG})$. Given a non-split link $L_1 \dots L_n$ in $f(G)$, we choose an orientation for each component. Because all of the bigons formed by directed edges bound disks in $f(\overline{DG})$, the cycles L_i^+ and L_i^- from $f(\overline{DG})$ may be thought of an element of $f(\Gamma(G))$, specifically, they are isotopic to L_i . Thus for any choice of orientation, $L_1^\pm \dots L_n^\pm$ is a distinct, non-split, consistently oriented link in $f(\overline{DG})$. Thus $f(\overline{DG})$ contains at least $k_n * 2^n$ such links.

Suppose that $L_1 \dots L_n$ is a consistently oriented n component link not constructed in the manner of the preceding paragraph. Suppose every L_i can be mapped to an element L'_i of $f(\Gamma(G))$ by forgetting orientation. Then the link $L'_1 \dots L'_n$ is split, so $L_1 \dots L_n$ is split as well. Thus, some L_i does not map to a cycle in $f(\Gamma(G))$. Then L_i is a bigon in $\Gamma(\overline{DG})$ formed from doubling an edge of G , and hence L_i bounds a disk in $f(\overline{DG})$. This implies that $L_1 \dots L_n$ is split as well.

Thus, the embedding $f(\overline{DG})$ has exactly $k_n * 2^n$ distinct consistently oriented non-split n component links, giving the bounds. \square

FIGURE 7. The bigons formed from L_1 .

Corollary 7.3. *For a graph G , $\overline{mnl}_n(\overline{DG}) \leq mnl_n(G) * 2^n$.*

Proof. Choose $f(G)$ to be an embedding that realizes $mnl_n(G)$, and apply Lemma 7.2. \square

Corollary 7.4. *If a graph G has an embedding that simultaneously realizes $mnl_n(G)$ for all n , then $\overline{mnl}(\overline{DG}) \leq \sum_n mnl_n(G) * 2^n$.*

Proof. Choose $f(G)$ to be an embedding that simultaneously realizes $mnl_n(G)$, and apply Lemma 7.2. \square

While we do not have an example, it seems that the condition on the simultaneous realization of $mnl_n(G)$ should not be vacuous. In [10], the second author found a graph G with $mnk(G) = 0$ and $mnl_3(G) = 0$, but such that every embedding of G contains either a 3-link or a nontrivial knot. Hence, the lower bounds of $mnk(G)$ and $mnl_3(G)$ cannot be simultaneously realized. Similar behavior is known for several of the graphs in Heawood family [12]. Thus, it seems possible that there is a graph G such that $mnl_n(G)$ and $mnl_m(G)$ cannot be simultaneously realized in any embedding of G .

Proposition 7.5. *Let G be a graph with $mnl(G) = 1$. Then $\overline{mnl}(\overline{DG}) = 4$.*

Proof. As G is intrinsically linked, it must contain one of the Petersen family graphs as a minor [16]. Thus, every embedding of G must contain a non-split 2 component link with non-zero linking number. So, $mnl_2(G) = 1$, and as $mnl(G) = 1$, $mnl_n(G) = 0$ for all $n > 2$. Note that an embedding $f(G)$ that realizes $mnl(G)$ also realizes $mnl_n(G)$ simultaneously for all n . Thus by Corollary 7.4 $\overline{mnl}(\overline{DG}) \leq 4$.

Let $f(\overline{DG})$ be an arbitrary embedding of \overline{DG} . We may obtain an embedding of G by deleting one of each pair of directed edges and ignoring orientations. Thus, in $f(\overline{DG})$ we may find two cycles L_1 and L_2 (possibly with inconsistent orientation) that have non-zero linking number.

Suppose L_1 is k edges in length. The edges of L_1 and the edges with opposite orientation can be thought of as subdividing the sphere into $k + 2$ regions, where the boundary of each region is a consistently oriented cycle (k bigons and 2 k -gons). See Figure 7. As L_2 has non-zero linking number with L_1 , it must have non-zero linking number with at least two of these consistently oriented cycles, call them α_1 and α_2 .

Similarly, we now consider the k' edges of L_2 and those of opposite orientation. They divide the sphere into $k'+2$ regions whose boundaries are consistently oriented cycles of \overline{DG} , so each α_i must have nonzero linking number with two consistent cycles β_{i1} and β_{i2} . This gives four distinct consistently oriented nonsplit links in $f(\overline{DG})$: $(\alpha_1, \beta_{11}), (\alpha_1, \beta_{12}), (\alpha_2, \beta_{21})$ and (α_2, β_{22}) .

Thus, every embedding of \overline{DG} must contain at least 4 distinct, consistently oriented non-split links. \square

Foisy-Howards-Rich show that if G has a linkless embedding, then \overline{DG} does as well in Theorem 3.6 of [11]. Combining that with Proposition 7.5, we have the following corollary.

Corollary 7.6. *If \overline{DG} is intrinsically linked, then $\overline{mnl}(\overline{DG}) \geq 4$.*

Note that this bound need not hold for arbitrary directed graphs. Specifically, if \overline{G} is the intrinsically linked digraph of Figure 1, the embedding f' of \overline{G} used in the proof of Theorem 2.1 has a single non-split consistently oriented link, hence $\overline{mnl}(\overline{G}) = 1$.

8. A REMARK ON LINKLESS EMBEDDINGS

In Section 2, we have already highlighted one important difference between intrinsic linking in undirected graphs and intrinsic linking in directed graphs, namely that the standard operation of taking minors preserves the property of having a linkless embedding for undirected graphs, but does not do so in the directed graph case. Here we highlight another important difference between the directed and undirected cases. Specifically, we consider the extremal problem of the maximum number of edges for a graph G or a digraph \overline{G} on n vertices that admits a linkless embedding. (Alternately, for $n > 6$, the minimal number of edges that forces G or \overline{G} to be intrinsically linked). For undirected graphs, the following is a corollary of standard results on graph minors.

Corollary 8.1. *The maximum number of edges in a graph G on n vertices that admits a linkless embedding is $O(n)$.*

Proof. There is a constant $c(6)$ such that if G has more than $c(6)n$ edges, then G has K_6 as a minor [14], [19]. Hence if G has more than $c(6)n$ edges, G is intrinsically linked. \square

In contrast, for directed graphs much denser examples may not be intrinsically linked as digraphs. Recall that a transitive tournament is an orientation of the complete graph K_n so that the edge ij is directed from i to j if $i < j$. Theorem 3.2 of [11] shows that a transitive tournament cannot be intrinsically linked as a digraph. We provide an even denser example below. Let $E(\overline{G})$ denote the edge set of \overline{G} and $|E(\overline{G})|$ the number of edges.

Theorem 8.2. *The maximum number of edges in a digraph on n vertices that admits an embedding with no pair of disjoint, consistently oriented cycles that form a nonsplit link is $O(n^2)$. Further, let c be the constant such that $|E(\overline{G})| > cn^2$ implies that \overline{G} is intrinsically linked as a digraph. Then $\frac{1}{2} < c \leq \frac{9}{10}$.*

Proof. Let $\overline{G'}$ be a transitive tournament on $n - 1$ vertices. Let \overline{G} be the graph formed by adding a vertex v and symmetric directed edges to and from v from and to

every vertex of $\overline{G'}$. As $\overline{G'}$ contains no consistently oriented cycles, any consistently oriented cycle in \overline{G} must pass through vertex v . Thus \overline{G} does not contain a pair of disjoint, consistently oriented cycles, so cannot be intrinsically linked as a digraph. By construction, \overline{G} has $\frac{(n+2)(n-1)}{2}$ edges. As $\overline{DK_n}$ has $2\binom{n}{2}$ edges, the maximum number of edges in a digraph is $O(n^2)$. Thus, the maximum number of edges in a digraph that admits a linkless embedding is $O(n^2)$. As $\frac{(n+2)(n-1)}{2} > \frac{n^2}{2}$, the example \overline{G} above shows that $c > \frac{1}{2}$.

We now address the upper bound for c . Let \overline{G} be a digraph with $|E(\overline{G})| > \frac{9}{10}n^2$. Let G' be an undirected graph on n vertices such that $w_iw_j \in E(G')$ if v_i and v_j of \overline{G} are connected with symmetric directed edges. As \overline{G} can be formed from $\overline{DK_n}$ by deleting fewer than $\frac{1}{10}n^2$ edges, at most $\frac{1}{10}n^2$ edges of \overline{G} are not part of a symmetric pair. Thus more than $\frac{8}{10}n^2$ edges of \overline{G} occur in symmetric pairs, so G' has more than $\frac{4}{5}\frac{n^2}{2}$ edges. By Turán's Theorem, G' contains K_6 as a subgraph, and by the construction of G' , this implies \overline{G} contains $\overline{DK_6}$ as a subdigraph. Therefore, \overline{G} must be intrinsically linked as a digraph. \square

A similar result holds for intrinsic knotting as $c(7)n$ edges on n vertices guarantee a K_7 minor ([14], [19]) and hence intrinsic knotting, but a transitive tournament with $\binom{n}{2}$ edges has no consistently oriented cycle, and hence cannot be intrinsically knotted as a digraph.

Examples of linkless or knotless digraphs with more edges than the examples discussed above may be possible. We leave this as an open question for future research.

REFERENCES

- [1] L. Abrams, B. Mellor, L. Trott, *Counting links and knots in complete graphs*, Tokyo J. Math, **36** No. 2 (2013) 429-458
- [2] J. H. Conway and C. McA. Gordon, *Knots and links in spatial graphs*, J. Graph Th. **7** (1983) 446-453
- [3] E. Flapan, J. Foisy, R. Naimi, and J. Pommersheim, *Intrinsically n -linked graphs*, J. Knot Theory Ramif. **10** No. 8 (2001) 1143-1154
- [4] E. Flapan, T. Mattman, B. Mellor, R. Naimi, R. Nikkuni *Recent developments in spatial graph theory*, preprint, arXiv:1602.08122v2
- [5] E. Flapan, R. Naimi, B. Mellor, *Intrinsic linking and knotting are arbitrarily complex*, Fundamenta Mathematicae, **201** (2008) 131-148
- [6] E. Flapan, R. Naimi, J. Pommershiem, *Intrinsically triple linked complete graphs*, Topol. Appl. **115** (2001) 239-246
- [7] T. Fleming, *Intrinsically linked graphs with knotted components*, J. Knot Theory Ramif. **21** No. 7 (2012)
- [8] T. Fleming and B. Mellor, *Counting links in complete graphs*, Osaka J. Math. **46** (2009), 173-201
- [9] J. Foisy, *Intrinsically knotted graphs*, J. Graph Th. **39** No. 3 (2002) 178-187
- [10] J. Foisy, *Graphs with a knot or 3-component link in every spatial embedding*, J. Knot Th. Ramif., **15** (2006) 1113-1118
- [11] J. Foisy, H. Howards, N. Rich *Intrinsic linking in directed graphs*, Osaka J. Math. **52** (2015) 817-831
- [12] R. Hanaki, R. Nikkuni, K. Taniyama, A. Yamazaki, *On intrinsically knotted or completely 3-linked graphs* Pac. J. Math. **252** (2011) 407-425
- [13] C. Jagger *Tournaments as strong subcontractions*, Discrete Math **176** (1997) 177-184
- [14] W. Mader, *Homomorphieigenschaften und mittlere Kantendichte von Graphen*, Mathematische Annalen **174** No. 4 (1967) 265-268

- [15] J. Nešetřil and R. Thomas, *A note on spatial representation of graphs* Commentat. Math. Univ. Carolinae **26** No. 4 (1985) 655-659
- [16] N. Robertson, P. Seymour, R. Thomas, *Sachs' linkless embedding conjecture*, J. Comb Theory Ser. B **64** (1995) 185-277
- [17] H. Sachs, *On spatial representations of finite graphs*, in: A. Hajnal, L. Lovasz, V.T. Sós (Eds.), Colloq. Math. Soc. János Bolyai, Vol. 37, North-Holland, Amsterdam, (1984) 649-662
- [18] K. Taniyama and A. Yasuhara, *Realization of knots and links in a spatial graph*, Topol. Appl. **112** (2001) 87-109
- [19] A. Thomason, *The extremal function for complete minors*, J. Comb Theory Ser. B **81** (2001) 318-338

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